# The inverse problem for heavy ion scattering in the framework of the algebraic scattering theory



1993 J. Phys. A: Math. Gen. 26 2047

(http://iopscience.iop.org/0305-4470/26/8/025)

J. Phys. A: Math. Gen. 26 (1993) 2047–2060. Printed in the UK

# The inverse problem for heavy ion scattering in the framework of the algebraic scattering theory

Andreas Zielke and Werner Scheid
Institut für Theoretische Physik der Justus-Liebig-Universität, Giessen, Federal Republic of Germany

Received 15 June 1992

**Abstract.** The inverse problem in the framework of the algebraic scattering theory (AST) is concerned with the derivation of potentials from an algebraic Hamiltonian connected with an algebraically determined $S$-matrix. We investigate the inverse problem for two kinds of dynamical symmetries, based on the groups $SO(1,3)$ and $SO(2,3)$, which have been proposed for an algebraic description of heavy ion scattering. Two different inversion schemes are presented yielding potentials for the $SO(1,3)$- and the $SO(2,3)$-problem. In the case of the $SO(2,3)$-symmetry a potential results which depends on the momentum operator and, therefore, can be interpreted as a non-local interaction.

## 1. Introduction

The algebraic scattering theory (AST), which was developed in the last decade by Alhassid, Iachello, Wu and others [1–9], provides an algebraic technique to determine the $S$-matrix for a given symmetry of the Hamiltonian of the scattering system. The AST has the advantage over the traditional scattering theory that the calculations involved are analytic and easy to carry out and that they result in a closed form for the $S$-matrix. Moreover, the algebraic treatment of coupled channel problems can be done without great numerical effort [10, 11].

Instead of a potential $V(r)$, which has to be known in the traditional treatment of scattering problems, the only information about the system in the algebraic theory is that its Hamiltonian has a dynamical symmetry. This is the case whenever the Hamiltonian $H$ can be written as a function $h$ of the Casimir invariants $C$ of a group $G : H = h(C)$ [7]. For one-dimensional scattering the groups $SO(2,1)$ and $SO(2,2)$ have been considered [4, 5, 12] and for three-dimensional scattering the groups $SO(1,3)$, $SO(2,3)$ and $SU(1,3)$ [6–10]. The determination of the $S$-matrix is completely algebraic without reference to a coordinate realization of the operators of the scattering group $G$. In the following we shall therefore distinguish the abstract Hamiltonian $H = h(C)$ and the algebraically derived $S$-matrix from their counterparts in traditional scattering theory by refering to them as algebraic Hamiltonian and algebraic $S$-matrix, respectively.

In the original formulation of the AST [1–9] it seemed to be possible to determine the algebraic $S$-matrix uniquely up to phase factors, which do not show up in the cross section. However, this result had to be reconsidered when we found that a coordinate realization of the algebraic Hamiltonian with an $SO(2,3)$ group structure led to a potential $V(r)$ whose $S$-matrix was not in agreement with the published

  

$SO(2,3)$ $S$-matrix [14,15]. In a subsequent re-examination of the AST [16] we showed that the original formulation of the AST is not the most general one. It has to be slightly modified in order to be in accordance with the potential $V(r)$ derived in [14,15]. The basic result of the re-examination is that in the algebraic theory two classes of $S$-matrices always have to be considered.

Up to now all explicit coordinate realizations of an algebraic Hamiltonian with a given dynamical symmetry have led to potentials reproducing only one of the two possible classes of $S$-matrices. This point is especially important in the case of the scattering groups $SO(1,3)$ and $SO(2,3)$, which are candidates for the description of modified Coulomb scattering [6–10].

The general algebraic $S$-matrix with $SO(2,3)$ symmetry has the form [8,16]

$$S^{\pm}_{lv} = e^{i\chi_{\pm} l}\frac{\Gamma(\frac{1}{2}(l+v+3/2 \pm if))\Gamma(\frac{1}{2}(l-v+3/2 \pm if))}{\Gamma(\frac{1}{2}(l+v+3/2 \mp if))\Gamma(\frac{1}{2}(l-v+3/2 \mp if))}\, e^{i\Phi_{\pm}(k,v)}\,. \tag{1}$$

Here, $l$ denotes the angular momentum, and the parameters $v$ and $f > 0$ are connected with the potential strength and the energy, respectively [8]. The phase factors $e^{i\chi_{\pm} l}$ and $e^{i\Phi_{\pm}(k,v)}$ cannot be fixed by means of group theory alone.

The $S$-matrices $S^{+}_{lv}$ constitute one class of matrices with the same pole structure, $S^{-}_{lv}$ the other one. The reason for the occurrence of two distinct classes is explained in detail in [16]. The Coulomb $S$-matrix

$$S^{\mathrm{Coul}}_{l} = \frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)} \tag{2}$$

can be obtained as a special case of $S^{+}_{lv}$: the constant $f$ has to be chosen to be the Sommerfeld parameter $\eta$, $f = \eta = (Z_1 Z_2 e^2)/k$ ($\hbar = m = 1$), and $v$ to be equal to $\frac{1}{2}$ [8,9]. The sign convention in the definition of $\eta$ is such that a positive $\eta$ corresponds to repulsive Coulomb scattering. The potential derived from (1) in [14,15], however, has an $S$-matrix which belongs to the class $S^{-}_{lv}$. To our knowledge no publication exists where coordinate-dependent potentials with a Coulomb tail or with an $S$-matrix belonging to the class $S^{+}_{lv}$ are derived from the algebraic $SO(2,3)$ Hamiltonian.

On the other hand it has been known for a long time that the group $SO(1,3)$ is the exact symmetry group for non-relativistic Coulomb scattering (cf e.g. [17]). Therefore, Coulomb scattering should be described by an AST with $SO(1,3)$ dynamical symmetry. The general $S$-matrix with $SO(1,3)$ symmetry is given by [6,8,16]

$$S^{\pm}_{l} = e^{i\chi_{\pm} l}\,\frac{\Gamma(l+1 \pm if)}{\Gamma(l+1 \mp if)} e^{i\Phi_{\pm}(k)}\,. \tag{3}$$

From this formula one sees that the Coulomb $S$-matrix (2) is in the class $S^{+}_{l}$.

The problem of the investigations presented in this paper is the derivation of potentials whose $S$-matrices belong to the class $S^{+}_{l}$ in the case of scattering with $SO(2,3)$ dynamical symmetry. The question is considered of whether these potentials have a long-range Coulomb tail plus short-range deviations from the pure Coulomb potential. Only if such potentials exist would the $SO(2,3)$-AST be a meaningful theory for an algebraic treatment of heavy ion scattering. Furthermore, we are interested to know if in the framework of the $SO(1,3)$-AST potentials can be found with $S$-matrices belonging to the class $S^{-}_{l}$ of (3).

To this end an inverse problem in the framework of the AST has to be solved for each symmetry. Starting from the algebraic $S$-matrix one obtains an expression for the algebraic Hamiltonian for which a coordinate realization has to be found such that one arrives at the usual Schrödinger equation. Then the potential can be read off directly.

Section 2 gives a short survey of some relevant results of the AST with $SO(1,3)$ and $SO(2,3)$ dynamical symmetry. In section 3 we show by means of two different inversion schemes that for the $SO(1,3)$-AST not only can the expected Coulomb potential be derived, but also a potential whose $S$-matrix belongs to the class $S_l^-$ of (3). Moreover, we give a solution for the $SO(1,3)$ inverse problem of modified Coulomb scattering. Section 4 deals with the inverse problem in the framework of the $SO(2,3)$-AST. We prove that the Coulomb potential appears as a special solution and point out some important consequences for the algebraic treatment of heavy ion scattering.

## 2. Short survey of the AST with $SO(1,3)$ and $SO(2,3)$ dynamical symmetry

### *2.1. The scattering group SO(1,3)*

The algebra of the group $SO(1,3)$ consists of six generators, $L_i$ and $K_j$ ($i,j \in 1,2,3$), satisfying the following commutation relations:

$$[K_i, K_j] = -\mathrm{i}\,\epsilon_{ijk} L_k \qquad [L_i, L_j] = \mathrm{i}\,\epsilon_{ijk} L_k \qquad [L_i, K_j] = \mathrm{i}\,\epsilon_{ijk} K_k\,. \tag{4}$$

$\boldsymbol{L} = (L_1, L_2, L_3)$ is the angular momentum operator. The relations (4) imply that the three operators $L_j$ form an $SO(3)$ subgroup of rotations and that the other three operators $(K_1, K_2, K_3) = \boldsymbol{K}$ behave like a vector under rotations.

A Casimir invariant of the group $SO(1,3)$ is given by

$$C^{SO(1,3)} = \boldsymbol{L}^2 - \boldsymbol{K}^2\,. \tag{5}$$

The general algebraic $S$-matrix with $SO(1,3)$ symmetry, given in (3), has one free parameter $f$. This parameter labels the eigenvalues of the Casimir invariant in the continuous series representations of $SO(1,3)$, which are used for the algebraic description of the scattering states [8, 6, 18]. One gets

$$C^{SO(1,3)} + 1 = -f^2 \tag{6}$$

when $C^{SO(1,3)}$ is acting on the scattering states.

In the $SO(1,3)$-AST the following ansatz is made for the algebraic Hamiltonian

$$H = h(-(C^{SO(1,3)} + 1))\,. \tag{7}$$

Since the Hamiltonian acting on scattering states can be substituted by the scattering energy, a relationship exists between the labelling parameter $f$ and the scattering energy $E$ [8]:

$$h(f^2) = E\,. \tag{8}$$

In order to obtain the Coulomb $S$-matrix one has to set

$$f = \eta = \frac{Z_1 Z_2 e^2}{k} \qquad \text{and} \qquad \chi_+ = \Phi_+ = 0 \tag{9}$$

in $S_l^+$ given in (3). $\eta$ denotes the real Sommerfeld parameter.

We remark that unless otherwise noted we consider only repulsive Coulomb scattering where $\eta > 0$. For attractive Coulomb scattering one has to set $f = -\eta$ in $S_l^-$, since the parameter $f$ is positive by convention.

Equations (7)–(9) allow the determination of the algebraic Hamiltonian which should describe Coulomb scattering in the $SO(1,3)$ theory. A solution is given by [8]

$$H = -\frac{\eta^2 E}{C^{SO(1,3)} + 1} = -\frac{(Z_1 Z_2 e^2)^2}{2\,(C^{SO(1,3)} + 1)}\,. \tag{10}$$

There are many other possible solutions of (7)–(9) which have not been mentioned in literature, e.g.

$$H = -\frac{(C^{SO(1,3)} + 1)E}{\eta^2}\,. \tag{11}$$

The Hamiltonian (10) is the only solution which does not depend explicitly on the energy.

It is important to note that there are four choices for $f$ which all lead to the same algebraic Hamiltonian (10): one can set $f = \eta$ (if $\eta > 0$) or $f = |\eta|$ in $S_l^+$ as well as $f = -\eta$ (if $\eta < 0$) or $f = |\eta|$ in $S_l^-$.

For the description of modified Coulomb scattering the parameter $f$ is split into a part $f_c = \eta$ describing pure Coulomb scattering and a part $f_s$ allowing for a short-range strong interaction plus absorption:

$$f = f_c + f_s\,.$$

Alhassid *et al* [10, 11, 9] showed that the qualitative features of heavy ion reaction cross sections can be reproduced in good quality if one chooses an $l$- and $k$-dependent parametrization of Woods–Saxon type for $f_s$:

$$f_s = \frac{f_R + \mathrm{i}\, f_I}{1 + e^{(l - l_0(k))/\Delta}}\,. \tag{12}$$

The imaginary part of $f_s$ describes absorption, $l_0$ has the meaning of a 'grazing' angular momentum and $\Delta$ is a measure of diffuseness.

For the collision of $^{16}O$ + $^{24}Mg$ at $E_{cm} = 27.8$ MeV a realistic calculation based on a generalization of the $SO(1,3)$-theory to four coupled channels has been carried out by Alhassid and Iachello [11].

### 2.2. *The scattering group SO(2,3)*

The algebra of the group $SO(2,3)$ consists of ten generators, $A_i$, $B_i$, $L_i$ and $V$, where $\boldsymbol{L} = (L_1, L_2, L_3)$ is the angular momentum operator, $\boldsymbol{A} = (A_1, A_2, A_3)$ and $\boldsymbol{B} = (B_1, B_2, B_3)$ are vectors under rotations while $V$ is a scalar. The commutation relations between $\boldsymbol{A}$, $\boldsymbol{B}$ and $V$ are

$$\begin{aligned} &[A_i, A_j] = [B_i, B_j] = -\mathrm{i}\,\epsilon_{ijk} L_k \qquad [A_i, B_j] = -\mathrm{i}\,\delta_{ij} V \\ &[A_i, V] = -\mathrm{i}\, B_i \qquad [B_i, V] = \mathrm{i}\, A_i\,. \end{aligned} \tag{13}$$

A Casimir invariant of the group $SO(2,3)$ is given by

$$C^{SO(2,3)} = \boldsymbol{L}^2 + V^2 - \boldsymbol{A}^2 - \boldsymbol{B}^2\,. \tag{14}$$

The algebraic procedure yields the $S$-matrix given in (1) which now has two free parameters, $f$ and $v$ [6,8,16]. Again, the parameter $f$ labels the continuous series representations of the Casimir invariant [6,8] and

$$C^{SO(2,3)} + \tfrac{9}{4} = -f^2 \,. \tag{15}$$

The parameter $v$ allows the variation of the potential strength as can be verified in any appropriate coordinate realization of the algebraic theory.

The following ansatz for the algebraic Hamiltonian is made:

$$H = h(-(C^{SO(2,3)} + \tfrac{9}{4})) \tag{16}$$

and (8) has to be fulfilled again. The Coulomb $S$-matrix appears as a special case of the general $SO(2,3)$ $S$-matrix $S^+_{lv}$ for the following values of the parameters in $S^+_{lv}$:

$$f = \eta \qquad v = \tfrac{1}{2} \qquad \text{and} \qquad \chi_+ = 0 \qquad \Phi_+ = 2f\ln 2 \tag{17}$$

where the theorem $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\,\pi^{1/2}\,\Gamma(2z)$ for the Gamma-functions has to be used.

As before, one obtains for the algebraic Hamiltonian

$$H = \frac{-\eta^2 E}{C^{SO(2,3)} + \frac{9}{4}} \tag{18}$$

or equivalently

$$H = -\frac{(C^{SO(2,3)} + \frac{9}{4})E}{\eta^2} \,. \tag{19}$$

It has the same form for one of the four choices, $f = \eta$ (if $\eta > 0$) or $f = |\eta|$ in $S^+_{lv}$ as well as $f = -\eta$ (if $\eta < 0$) or $f = |\eta|$ in $S^-_{lv}$.

The algebraic description of modified Coulomb scattering makes use of the fact that the $SO(2,3)$ $S$-matrix (1) has one additional parameter $v$ compared to the $SO(1,3)$ $S$-matrix (3). Now the potential strength parameter $v$ allows for short-range deviations from pure Coulomb scattering. The parameter $f$ remains fixed while $v$ is taken to be different from $\frac{1}{2}$ and parametrized as a function of $l$ and $k$ [8,9]:

$$v = v(l,k) = \frac{v_{\mathrm{R}} + \mathrm{i}\, v_{\mathrm{I}}}{1 + \mathrm{e}^{(l-l_0(k))/\Delta}} \,. \tag{20}$$

A fit of the four parameters $v_{\mathrm{R}}$, $v_{\mathrm{I}}$, $\Delta$, and $l_0$ in the $S$-matrix (1) to experimentally measured cross sections for the scattering of $^{16}$O on $^{28}$Si gave good results, comparable to those of optical model fits [6,9].

### 3. Solutions of the $SO(1,3)$ inverse problem

In the AST the algebraic $S$-matrix is determined without knowledge of the potentials, which are needed for the calculation of the $S$-matrix in traditional scattering theory. In order to obtain the potentials belonging to a particular $S$-matrix one can follow two routes. The first one is to use one of the mostly numerical inversion methods of the inverse scattering theory [19,20,13]. The other one, to be followed here, starts from the algebraic Hamiltonian with $SO(1,3)$ symmetry, which assumes a particular form once the free parameters of the $S$-matrix are fixed (cf (10) and (11)). The idea of the inversion method in the context of the AST is to search for a realization of the group-theoretical Hamiltonian in suitable coordinates such that one obtains the usual form of the Schrödinger equation.

Among many possible realizations of the $SO(1,3)$ generators we consider the following one which is particularly suitable for our purpose:

$$\boldsymbol{L} = \boldsymbol{R} \times \boldsymbol{P} \qquad \boldsymbol{K} = \frac{1}{\sqrt{2E}}\left(\frac{1}{2}\boldsymbol{R}P^2 - \boldsymbol{P}(\boldsymbol{R}\boldsymbol{P})\right) + \frac{\sqrt{2E}}{2}\boldsymbol{R}\,. \tag{21}$$

Here and in the following the realization coordinates $\boldsymbol{R}$ and $\boldsymbol{P}$ are conjugate to each other: $[R_i, P_j] = \mathrm{i}\,\delta_{ij}$. In our units $\hbar = m = 1$. For the Casimir invariant $C^{SO(1,3)} + 1 = \boldsymbol{L}^2 - \boldsymbol{K}^2 + 1$ we get

$$C^{SO(1,3)} + 1 = -\frac{1}{8E}R^2P^4 + \frac{1}{4E}(\mathrm{i}\boldsymbol{R}\boldsymbol{P})P^2 + \frac{1}{2}R^2P^2 - \frac{1}{2}\mathrm{i}\boldsymbol{R}\boldsymbol{P} + \frac{1}{4E}P^2 - \frac{E}{2}R^2 - \frac{1}{2}\,. \tag{22}$$

Proceeding from this specific realization of the $SO(1,3)$ Casimir operator we have found two different algebraic inversion schemes leading to two distinct classes of potentials.

The first inversion scheme starts from the observation that in the realization of (22) the momentum operator $\boldsymbol{P}$ appears in fourth order, whereas the position operator $\boldsymbol{R}$ appears only up to second order. As we want to get a Schrödinger equation in the end, where the momentum appears only up to second order, we make a canonical transformation

$$\boldsymbol{R} \to \frac{1}{\sqrt{2E}}\boldsymbol{P} \qquad \boldsymbol{P} \to -\sqrt{2E}\boldsymbol{R}\,.$$

A canonical transformation does not change the commutation relations of the transformed $SO(1,3)$ operators. With the substitutions

$$P^2 = -\frac{\partial^2}{\partial R^2} - \frac{2}{R}\frac{\partial}{\partial R} + \frac{\boldsymbol{L}^2}{R^2} \qquad \boldsymbol{P} = -\mathrm{i}\,\nabla$$

we obtain for the Casmimir operator (22)

$$\begin{aligned} C^{SO(1,3)} + 1 = {} & \frac{1}{4}(R^2-1)^2\frac{\partial^2}{\partial R^2} + \frac{1}{2R}(4R^4 - 5R^2 + 1)\frac{\partial}{\partial R} \\ & - \frac{(R^2-1)^2}{4R^2}\boldsymbol{L}^2 + 3R^2 - 2\,. \end{aligned} \tag{23}$$

Following a procedure which is described in detail in [14] we make two further transformations which lead to the Schrödinger equation in the end:

(i) a similarity transformation of the operators with $T = T(R)$

$$C^{SO(1,3)} \longrightarrow T^{-1}C^{SO(1,3)}T$$

(ii) a transformation of the coordinate $R$: $r = g(R)$.

The latter transformation connects the representation coordinate $R$ with the physical radial coordinate $r$.

Introducing the wavenumber $k = \sqrt{2E}$ and choosing

$$r = g(R) = \frac{2|\eta|}{k} \coth^{-1} R \tag{24}$$

and

$$T(r) = r \sinh^2\left(\frac{rk}{2|\eta|}\right) \tanh\left(\frac{rk}{2|\eta|}\right) \tag{25}$$

we obtain

$$-\frac{k^2}{\eta^2}\left(C^{SO(1,3)} + 1\right) = -\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{k^2}{\eta^2}\frac{L^2}{\sinh^2(rk/|\eta|)} \,. \tag{26}$$

This realization of the Casimir operator cannot be used directly in the expression (10) for the Hamiltonian because the Casimir invariant appears in the denominator. We first have to multiply both sides of (10) from the left by $-(C^{SO(1,3)}+1)/\eta^2$ and obtain

$$-\frac{C^{SO(1,3)} + 1}{\eta^2} H = E \,.$$

Then we substitute $H$ by its eigenvalue $E$ and vice versa. The last manipulation is allowed, since in the AST the operators always act on states of fixed energy $E$. Thus we get

$$-\frac{C^{SO(1,3)} + 1}{\eta^2} E = H$$

which is equal to the expression (11).

Using (26) and $E = k^2/2$ we finally obtain

$$H = \frac{1}{2}\left(-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{k^2}{\eta^2}\frac{L^2}{\sinh^2(rk/|\eta|)}\right) . \tag{27}$$

Equation (27) has the form of a three-dimensional Schrödinger equation which separates in spherical coordinates. We have

$$H = \frac{p^2}{2} + V(r)$$

where

$$\frac{V(r)}{E/\eta^2} = -l(l+1)\left(\frac{1}{(rk/|\eta|)^2} - \frac{1}{\sinh^2(rk/|\eta|)}\right) . \tag{28}$$

The potential $V(r)$ is of Pöschl–Teller type and depends on the energy $E$ and the angular momentum $l$. It is finite at the origin and goes like $1/r^2$ for large values of the radial coordinate $r$.

The corresponding $S$-matrix can be calculated analytically by means of traditional scattering theory:

$$S_l = \mathrm{e}^{\mathrm{i}l\pi} \frac{\Gamma(l+1-\mathrm{i}|\eta|)}{\Gamma(l+1+\mathrm{i}|\eta|)} \mathrm{e}^{\mathrm{i}\pi - 2\mathrm{i}\arg\Gamma(1-\mathrm{i}|\eta|)} . \tag{29}$$

We observe that it belongs to the class $S_l^-$ of algebraic $SO(1,3)$ $S$-matrices ($f = |\eta|$, cf (3)). In the chosen realization the algebraically undetermined phase factors assume definite values:

$$\chi_- = \pi \qquad \text{and} \qquad \Phi_\pm(k) = \pi - 2\arg\Gamma(1-\mathrm{i}|\eta|) .$$

So far the inversion has been restricted to the choice $f = \eta$ or $f = |\eta|$ in the algebraic $S$-matrix, i.e. to an algebraic Hamiltonian of the form (10). In the general case the Hamiltonian reads

$$H = -\frac{f^2 E}{C^{SO(1,3)} + 1} . \tag{30}$$

Writing $f$ instead of $|\eta|$ in the transformations (24) and (25) and following the steps (26) to (28) one obtains the potential

$$\frac{V(r)}{E/f^2} = -l(l+1)\left(\frac{1}{(rk/f)^2} - \frac{1}{\sinh^2(rk/f)}\right) . \tag{31}$$

In the second inversion scheme we start with the realization (22) of the $SO(1,3)$ Casimir invariant. We found that (22) can be written in the form

$$C^{SO(1,3)} + 1 = -\left[\frac{R}{\sqrt{2E}}\left(\frac{\boldsymbol{P}^2}{2} - E\right)\right]^2 . \tag{32}$$

The Casimir invariant is a constant in the irreducible representation space of scattering states with fixed scattering energy $E$. From (10) or (11) we get

$$C^{SO(1,3)} + 1 = -\eta^2 . \tag{33}$$

Comparing the right-hand sides of (32) and (33) we get a solution of the form

$$\frac{R}{\sqrt{2E}}\left(\frac{\boldsymbol{P}^2}{2} - E\right) = -\eta = -\frac{Z_1 Z_2 e^2}{\sqrt{2E}} . \tag{34}$$

Multiplying both sides by $\sqrt{2E}/R$ from the left and substituting $E$ by $H$ we obtain

$$H = \frac{\boldsymbol{P}^2}{2} + \frac{Z_1 Z_2 e^2}{R} . \tag{35}$$

Thus, in the second inversion scheme the representation coordinate $R = |\boldsymbol{R}|$ is identical with the physical radial coordinate and one obtains the Coulomb potential as one possible solution. Clearly, the $S$-matrix corresponding to the Hamiltonian (35) belongs to the class $S_l^+$ of algebraic $SO(1,3)$ $S$-matrices (for $\eta > 0$).

Another solution of (32) and (33) is given for

$$\frac{R}{\sqrt{2E}}\left(\frac{\boldsymbol{P}^2}{2} - E\right) = \eta .$$

In this case one obviously gets a Coulomb potential which is attractive for $Z_1Z_2e^2 > 0$. This amounts to a sign convention for $\eta$ opposite to the one adopted in section 1. Consequently, the corresponding $S$-matrix belongs to the class $S_l^-$ (for $\eta > 0$).

For $f = \eta + f_s$ modified Coulomb scattering should be described in the $SO(1,3)$-AST. In this case one has to solve (32) and

$$C^{SO(1,3)} + 1 = -f^2 = -(\eta + f_s)^2 .$$

By the same arguments as above we get

$$H = \frac{P^2}{2} + \frac{Z_1Z_2e^2}{R}\left(1 + \frac{f_s}{\eta}\right) \tag{36}$$

instead of (35). It turns out that a constant parameter $f$ leads only to a modified Coulomb potential and not to a short-range interaction. However, if one chooses an $l$-dependent parametrization for $f$, as e.g. in (12), it should be possible to simulate a short-range interaction.

In summary, in the $SO(1,3)$-AST two classes of potentials can be described, those of Pöschl–Teller type and—as expected—those of Coulomb type. Other realizations or different inversion schemes might yield further potentials.

## 4. Solutions of the $SO(2,3)$ inverse problem

The simplest approach to the derivation of potentials in the $SO(2,3)$-AST goes along the same lines as in the $SO(1,3)$-AST. First we have to look for realizations of the generators of the $SO(2,3)$ algebra in appropriate coordinates, then we try to employ the same two inversion schemes as in the preceding section. It is easy to see that neither of the two inversion schemes can be applied when $P$ and $R$ appear in more than fourth order in the realization of the Casimir invariant.

We found an appropriate realization of the $SO(2,3)$ operators, which is constructed from an $SO(2,3)$ realization similar to the realization given in (21). Introducing additional non-commuting operators $v_1$, $v_2$, and $v_3$ we consider the following representation:

$$\begin{aligned} A &= v_1\left(-\mathcal{P}(\mathcal{R}\mathcal{P}) + \mathcal{R} - \frac{3\mathrm{i}}{2}\mathcal{P}\right) + \{v_2, v_3\}\,\frac{\mathcal{P}}{2} \\ B &= v_2\left(-\mathcal{P}(\mathcal{R}\mathcal{P}) + \mathcal{R} - \frac{3\mathrm{i}}{2}\mathcal{P}\right) - \{v_1, v_3\}\,\frac{\mathcal{P}}{2} \\ L &= \mathcal{R} \times \mathcal{P} \qquad V = v_3 \end{aligned} \tag{37}$$

where

$$[v_1, v_2] = 0 \qquad [v_1, v_3] = -\mathrm{i}v_2 \qquad [v_2, v_3] = \mathrm{i}v_1 .$$

Here $\{\ ,\ \}$ denotes the anticommutator, i.e. $\{a, b\} = ab + ba$. One can think of $v_1$, $v_2$, and $v_3$ as operators connected with an additional coordinate $\chi$ related to the operator $V$. For instance, one may set $v_3 = -\mathrm{i}\partial/\partial\chi$, $v_1 = \cos\chi$ and $v_2 = \sin\chi$

in (37). Of course, one has to require that the Casimir invariant $C^{SO(2,3)}$ depend only on $v_3 = V$. We get

$$C^{SO(2,3)} + \frac{9}{4} = -\left(-\mathcal{P}(\mathcal{R}\mathcal{P}) + \mathcal{R} - \frac{3\mathrm{i}}{2}\mathcal{P}\right)^2 + (\mathcal{R}\times\mathcal{P})^2 + 1 - \left(v_3^2 - \frac{1}{4}\right)(\mathcal{P}^2 - 1). \tag{38}$$

In order to employ the first inversion scheme one has to introduce a transformation so that the momentum appears only quadratic in the realization of $C^{SO(2,3)}$. This can be achieved by making the canonical transformation in (38):

$$\mathcal{R} \longrightarrow -\mathcal{P} \qquad \mathcal{P} \longrightarrow \mathcal{R}.$$

Writing now all operators in configuration space we obtain a coordinate representation for $C^{SO(2,3)} + \frac{9}{4}$, which after a similarity transformation with $T(\mathcal{R}) = \exp(-\frac{1}{2}\mathrm{i}\pi\mathcal{R}\partial/\partial\mathcal{R})$ takes the form

$$C^{SO(2,3)} + \frac{9}{4} = (\mathcal{R}+1)^2\left(-\frac{\partial^2}{\partial\mathcal{R}^2} - \frac{2}{\mathcal{R}}\frac{\partial}{\partial\mathcal{R}}\right) + \frac{\mathcal{R}^2+1}{\mathcal{R}^2}L^2 + 1 + \left(v^2 - \frac{1}{4}\right)(1+\mathcal{R}^2). \tag{39}$$

Here, the operator $v_3 = V$ has been substituted by its eigenvalue $v$. The transformation with $T(\mathcal{R})$ belongs to the class of scaling transformations $T = \mathrm{e}^{-\mathrm{i}\alpha\mathcal{R}\mathcal{P}}$, $P = -\mathrm{i}\,\partial/\partial\mathcal{R}$, which introduce a rescaling of $\mathcal{R}$ and $\mathcal{P}$: $\mathcal{R} \to \mathrm{e}^{\alpha}\mathcal{R}$, $\mathcal{P} \to \mathrm{e}^{-\alpha}\mathcal{P}$. As outlined in section 3 one continues by connecting the realization coordinate $\mathcal{R}$ with the physical radial coordinate $r$

$$r = g(\mathcal{R}) = \frac{|\eta|}{k}\,\mathrm{artanh}\mathcal{R} \tag{40}$$

followed by a second similarity transformation with

$$T(r) = \frac{r}{\sinh(rk/|\eta|)}. \tag{41}$$

Both transformations, (40) and (41), are chosen in such a way that the transformed $SO(2,3)$ Casimir operator has the form of a Schrödinger Hamiltonian. We get

$$-\frac{k^2}{\eta^2}\left(C^{SO(2,3)} + \tfrac{9}{4}\right) = -\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{k^2}{\eta^2}\frac{L^2}{\sinh^2(rk/|\eta|)} - \frac{k^2}{\eta^2}\frac{v^2 - \frac{1}{4}}{\cosh^2(rk/|\eta|)}. \tag{42}$$

The realization of the algebraic Hamiltonian is given by (cf (19))

$$H = \frac{1}{2}\left(-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{k^2}{\eta^2}\frac{L^2}{\sinh^2(rk/|\eta|)} - \frac{k^2}{\eta^2}\frac{v^2 - \frac{1}{4}}{\cosh^2(rk/|\eta|)}\right). \tag{43}$$

The potential can be read off directly:

$$H = p^2/2 + V(r)$$

where

$$\frac{V(r)}{E/\eta^2} = -l(l+1)\left(\frac{1}{(rk/|\eta|)^2} - \frac{1}{\sinh^2(rk/|\eta|)}\right) - \frac{v^2-\frac{1}{4}}{\cosh^2(rk/|\eta|)}\,. \tag{44}$$

We observe that the first, $l$-dependent term of the potential (44) is identical with the potential (28) derived in the $SO(1,3)$-AST. The second term contributes only for $v^2 \neq \frac{1}{4}$. A variation of the free parameter $v$ allows alterations to the shape of the potential. For this reason $v$ is called potential strength parameter in the AST [5,8,9].

It is surprising to note that the potential (44) is the same one we derived in an earlier paper [14] from a different realization of the $SO(2,3)$ generators (cf equations (6)–(19) of [14]). We found yet another suitable realization of the $SO(2,3)$ generators:

$$\begin{aligned} \boldsymbol{A} &= (v_1 \mp \mathrm{i}\, v_2)\left(\frac{\mathcal{R}}{2}\mathcal{P}^2 + \sigma\mathcal{R} - \frac{\mathrm{i}}{2}\mathcal{P}\right) \pm \mathrm{i}\, v_2\mathcal{R} + (\{v_2, v_3\} \pm \mathrm{i}\,\{v_1, v_3\})\,\frac{\mathcal{P}}{2} \\ \boldsymbol{B} &= (v_2 \pm i\, v_1)\left(\frac{\mathcal{R}}{2}\mathcal{P}^2 + \sigma\mathcal{R} - \frac{\mathrm{i}}{2}\mathcal{P}\right) \mp \mathrm{i}\, v_1\mathcal{R} - (\{v_1, v_3\} \mp \mathrm{i}\,\{v_2, v_3\})\,\frac{\mathcal{P}}{2} \\ \boldsymbol{L} &= \mathcal{R}\times\mathcal{P} \qquad V = v_3 \end{aligned} \tag{45}$$

for which the first inversion scheme yields again the potential (44). The parameter $\sigma$ may be chosen arbitrarily. The $S$-matrix corresponding to the potential (44) can be calculated analytically in the framework of traditional scattering theory:

$$S_{lv} = \mathrm{e}^{\mathrm{i}l\pi}\,\frac{\Gamma(\frac{1}{2}(l+v+\frac{3}{2}-\mathrm{i}|\eta|))\Gamma(\frac{1}{2}(l-v+\frac{3}{2}-\mathrm{i}|\eta|))}{\Gamma(\frac{1}{2}(l+v+\frac{3}{2}+\mathrm{i}|\eta|))\Gamma(\frac{1}{2}(l-v+\frac{3}{2}+\mathrm{i}|\eta|))}\,\mathrm{e}^{\mathrm{i}\pi-2\mathrm{i}(\arg\Gamma(1-\mathrm{i}|\eta|)+|\eta|\ln 2)}\,. \tag{46}$$

It belongs to the class $S_l^-$ of algebraic $S$-matrices ($f = |\eta|$, cf (1)) , where the algebraically undetermined phase factors take the values

$$\chi_- = \pi \qquad \text{and} \qquad \Phi_-(k) = \pi - 2\,(\arg\Gamma(1-\mathrm{i}|\eta|) + |\eta|\ln 2)\;.$$

The second inversion scheme cannot be applied directly to the realization (38) of the $SO(2,3)$ Casimir invariant. As shown in section 3 we know how to solve the inverse problem for $H^{SO(1,3)} = -\eta^2 E/(C^{SO(1,3)}+1)$ with the Casimir invariant given by (32). The solution makes use of the particular form of the realization of the $SO(1,3)$ Casimir invariant. Now we want to solve the $SO(2,3)$ inverse problem where

$$H^{SO(2,3)} = \frac{-\eta^2 E}{C^{SO(2,3)}+\frac{9}{4}}$$

and

$$C^{SO(2,3)} + \frac{9}{4} = -\left(-\mathcal{P}(\mathcal{R}\mathcal{P}) + \mathcal{R} - \frac{3\mathrm{i}}{2}\mathcal{P}\right)^2 + (\mathcal{R}\times\mathcal{P})^2 + 1 - \left(v^2 - \frac{1}{4}\right)(\mathcal{P}^2 - 1)\,. \tag{47}$$

Furthermore, we would like to show that it is possible to obtain a pure Coulomb potential for $v = \frac{1}{2}$. Therefore, we look for a set of transformations which converts the particular realization of (47) with $v^2 = \frac{1}{4}$ into the form of the realization of (32). Beforehand, it is not clear that such a set of transformations exists. We could not find any $\mathcal{R}$-dependent transformations in configuration space leading to the desired result. Instead, one has to consider $\mathcal{P}$-dependent transformations in momentum space. Making a scaling transformation with $T = \exp(\frac{\mathrm{i}}{2}(\ln 2E)\, RP)$ in (32) and substituting $R$ by $\mathrm{i}\nabla_P$ and $R^2$ by

$$\left(-\frac{\partial^2}{\partial P^2} - \frac{2}{P}\frac{\partial}{\partial P} + \frac{L^2}{P^2}\right)$$

in (32) and in (47), one obtains the following expressions:

$$C^{SO(1,3)} + 1 = \frac{1}{4}(1-P^2)^2\frac{\partial^2}{\partial P^2} + \frac{1}{2P}(4P^4 - 5P^2 + 1)\frac{\partial}{\partial P} - \frac{(1-P^2)^2}{4P^2}L_P^2 + 3P^2 - 2 \tag{48}$$

$$C^{SO(2,3)} + \frac{9}{4} = (1-\mathcal{P}^2)^2\frac{\partial^2}{\partial \mathcal{P}^2} + \frac{1}{\mathcal{P}}(1-\mathcal{P}^2)(2-11\mathcal{P}^2)\frac{\partial}{\partial \mathcal{P}} + \frac{\mathcal{P}^2-1}{\mathcal{P}^2}L_{\mathcal{P}}^2 + \frac{99}{4}\mathcal{P}^2 - \frac{25}{2} - \left(\frac{1}{4} - v^2\right)(1-\mathcal{P}^2)\,. \tag{49}$$

Now the right-hand side of (49) can be transformed to the form of the right-hand side of (48) by means of the following $\mathcal{P}$-dependent transformations:

(i) a similarity transformation with

$$T = T(\mathcal{P}) = \exp\left(\frac{\frac{9}{4}}{1-\mathcal{P}^2}\right) \tag{50}$$

(ii) a transformation of the momentum coordinate

$$P = g(\mathcal{P}) = \begin{cases} -\dfrac{1+\sqrt{1-\mathcal{P}^2}}{\mathcal{P}} & \text{for } \mathcal{P} \in [-1, 0) \\ \dfrac{1-\sqrt{1-\mathcal{P}^2}}{\mathcal{P}} & \text{for } \mathcal{P} \in [0, 1] \end{cases} \tag{51}$$

(iii) a similarity transformation with

$$\tilde{T} = \tilde{T}(P) = (P-1)^2\,. \tag{52}$$

It turns out that $P$ is the physical momentum scaled by a factor of $1/\sqrt{2E}$.

Applying the transformations (i)–(iii) one obtains for the $SO(2,3)$ Casimir invariant

$$C^{SO(2,3)} + \frac{9}{4} = \frac{1}{4}(1-P^2)^2\frac{\partial^2}{\partial P^2} + \frac{1}{2P}(4P^4 - 5P^2 + 1)\frac{\partial}{\partial P} - \frac{(1-P^2)^2}{4P^2}L_P^2 + 3P^2 - 2 + \left(v^2 - \frac{1}{4}\right)\left(\frac{1-P^2}{1+P^2}\right)^2\,. \tag{53}$$

After a further scaling transformation with $T = \exp\left(-\frac{1}{2}\mathrm{i}(\ln 2E)\, RP\right)$ and the identification of $P$ with the physical momentum, the $SO(2,3)$ Casimir operator (53) reads in abstract Hilbert space

$$C^{SO(2,3)} + \frac{9}{4} = -\left[\frac{R}{\sqrt{2E}}\left(\frac{\boldsymbol{P}^2}{2} - E\right)\right]^2 + \left(v^2 - \frac{1}{4}\right)\left(\frac{E - P^2/2}{E + P^2/2}\right)^2 . \tag{54}$$

Thus, we have found a realization of $C^{SO(2,3)} + \frac{9}{4}$, which for $v = \frac{1}{2}$ assumes the same form as the realization (32) of the $SO(1,3)$ term $C^{SO(1,3)} + 1$. For $v = \frac{1}{2}$ one can apply the second inversion scheme to the realization (54) and obtains the Coulomb potential as in section 3:

$$H = \frac{\boldsymbol{P}^2}{2} + \frac{Z_1 Z_2 e^2}{R} . \tag{55}$$

For $v \neq \frac{1}{2}$ it is possible to derive an operator expression for the potential $V$. We first make the ansatz

$$V = \frac{Z_1 Z_2 e^2}{R} + \sqrt{2E}\mathcal{O} \tag{56}$$

where $\mathcal{O}$ is a yet undetermined operator. Then we substitute the term $(E - \boldsymbol{P}^2/2)$ acting on the scattering states by $V$. This allows us to write

$$\left[\frac{R}{\sqrt{2E}}\left(\frac{\boldsymbol{P}^2}{2} - E\right)\right]^2 = \eta^2 + \left(\eta + \frac{R}{\sqrt{2E}}\left(E - \frac{\boldsymbol{P}^2}{2}\right)\right) R\mathcal{O} . \tag{57}$$

Using (15) with $f = \eta$ we determine the operator $\mathcal{O}$ by comparing (57) with (54). We finally obtain

$$V = V(R, \boldsymbol{P}) = \frac{Z_1 Z_2 e^2}{R} + \left(v^2 - \frac{1}{4}\right)\frac{2E}{R} \\ \times \left(Z_1 Z_2 e^2 + R\left(E - \frac{\boldsymbol{P}^2}{2}\right)\right)^{-1}\left(\frac{E - P^2/2}{E + P^2/2}\right)^2 . \tag{58}$$

For $v \neq \frac{1}{2}$ the potential (58) becomes non-local. The non-locality is introduced through a complicated momentum- and position-dependent operator which is not easy to handle. Work on an interpretation of the non-local part as well as on an analytical calculation of the $S$-matrix in the framework of the traditional scattering theory is in progress.

The occurrence of non-local potentials, which has also been conjectured by Alhassid (cf p 472 of [9]), is in agreement with the conclusions drawn from a numerical inversion of the phase shifts calculated from a given parametrization of the $SO(2,3)$ $S$-matrix [13]. There, the energy dependence and rapid oscillations of the numerically determined local potential pointed to an underlying non-local interaction. This assumption was corroborated in a subsequent paper [21] where the local, energy-dependent potential found in [13] was connected to a smooth, local and energy-independent potential plus a non-local interaction.

## 5. Summary and Conclusions

In the AST with $SO(1,3)$ or $SO(2,3)$ dynamical symmetry the respective general $S$-matrices contain the Coulomb $S$-matrix as a special case. For both groups the general algebraic $S$-matrix encompasses two classes of $S$-matrices with different pole structures. However, in the algebraic theory the potentials corresponding to these $S$-matrices are not known.

In this paper two different inversion schemes have been presented, which provide the connection of the algebraic $S$-matrix with underlying potentials. Corresponding to the two inversion schemes one gets two different types of potentials, each type reproducing an $S$-matrix belonging to one class of algebraic $S$-matrices. The potentials are of Pöschl–Teller type for the first and of Coulomb type for the second inversion scheme.

The $SO(1,3)$ inverse problem turns out to be a special case of the $SO(2,3)$ inverse problem and can be solved completely. In the case of the $SO(2,3)$ group structure the second inversion scheme yields a pure Coulomb potential for the value $v = \frac{1}{2}$ of the potential strength parameter. For $v \neq \frac{1}{2}$ an additional momentum-dependent term comes into play which corresponds to a non-local potential and necessitates further investigations.

The most important result of this paper is that we have found for the first time a procedure to relate the algebraic $SO(1,3)$ and $SO(2,3)$ $S$-matrices with Coulomb potentials via an appropriate realization of the group-theoretical Hamiltonian. This proves that cross sections, calculated by using the algebraic $SO(1,3)$ and $SO(2,3)$ $S$-matrices with parameters fitted to experimentally measured cross sections, bear a physical meaning in the sense that they can be traced back to the scattering of particles in a Coulomb potential modified by a non-local interaction.

## References


[1] Alhassid Y, Gürsey F and Iachello F 1983 *Phys. Rev. Lett.* **50** 873
[2] Alhassid Y, Gürsey F and Iachello F 1983 *Ann. Phys.* **148** 346
[3] Frank A and Wolf K B 1984 *Phys. Rev. Lett.* **52** 1737
[4] Alhassid Y, Engel J and Wu J 1984 *Phys. Rev. Lett.* **53** 17
[5] Alhassid Y, Gürsey F and Iachello F 1986 *Ann. Phys.* **167** 181
[6] Wu J 1985 *PhD Thesis* Yale University
[7] Alhassid Y, Iachello F and Wu J 1986 *Phys. Rev. Lett.* **56** 271
[8] Wu J, Iachello F and Alhassid Y 1987 *Ann. Phys.* **173** 68
[9] Alhassid Y 1986 *Nuclear Structure, Reactions and Symmetries* vol 1, ed R A Meyer and V Paar (Singapore: World Scientific) p 491
Iachello F 1986 *Nuclear Structure, Reactions and Symmetries* vol 1 ed R A Meyer and V Paar (Singapore: World Scientific) p 455
[10] Alhassid Y, Iachello F and Shao B 1987 *Phys. Lett.* **201** B 183
[11] Alhassid Y and Iachello F 1989 *Nucl. Phys.* A **501** 585
[12] Wu J, Alhassid Y and Gürsey F 1989 *Ann. Phys.* **196** 163
[13] Maass R, May K E and Scheid W 1989 *Phys. Rev.* C **39** 1201
[14] Zielke A, Maass R, Scheid W and Weaver O L 1990 *Phys. Rev.* A **42** 1358
[15] Zielke A 1990 *Diploma Thesis* Justus-Liebig-Universität Giessen (unpublished)
[16] Zielke A and Scheid W 1992 *J. Phys. A: Math. Gen.* **25** 1383
[17] Zwanziger D 1967 *J. Math. Phys.* **8** 1858
[18] Naimark M A 1964 *Linear Representations of the Lorentz Group* (New York: Pergamon)
[19] Chadan K and Sabatier P C 1989 *Inverse Problems in Quantum Scattering Theory* (New York: Springer)
[20] May K E and Scheid W 1989 *Nucl. Phys.* A **499** 200
[21] Apagyi B, May K E, Häuser T and Scheid W 1990 *J. Phys. G: Nucl. Part. Phys.* **16** 451